Online Appendix: Identifying Socially Disruptive Policies

Eric Auerbach[*]   Yong Cai [†]

June 26, 2023

B Auxiliary lemmas

Lemma B1 (Lusin): For any measurable $f : [0,1]^2 \to \mathbb{R}$ and $\epsilon > 0$ there exists a compact $E_\epsilon \subseteq [0,1]^2$ with Lebesgue measure at least $1-\epsilon$ such that f is continuous when restricted to E_ϵ. See Dudley (2002) Theorem 7.5.2.

Lemma B2 (Spectral): Let $f : [0,1]^2 \to \mathbb{R}$ be a bounded symmetric measurable function and $T_f : L_2([0,1]) \to L_2([0,1])$ the associated integral operator $(T_f g)(u) = \int f(u,\tau)g(\tau)d\tau$. T_f admits the spectral decomposition $f(u,v) = \sum_{r=1}^{\infty} \lambda_r \phi_r(u), \phi_r(v)$ in the sense that $(T_f g)(u) = \int f(u,\tau)g(\tau)d\tau = \sum_{r=1}^{\infty} \lambda_r \phi_r(u) \int \phi_r(\tau)g(\tau)d\tau$ for any $g \in L_2([0,1])$. Each (λ_r, ϕ_r) pair satisfies $\int f(u,\tau)\phi_r(\tau)d\tau = \lambda_r \phi_r(u)$ where $\{\lambda_r\}_{r=1}^{\infty}$ is a multiset of bounded real numbers with 0 as its only limit point and $\{\phi_r\}_{r=1}^{\infty}$ is an orthogonal basis of $L_2([0,1])$. See Birman and Solomjak (2012) equation (5) preceding Theorem 4 in Chapter 9.2.

The spectral decomposition in Lemma B2 is different than the usual one for matrices. That is, if Y is an $N \times N$ dimensional symmetric real-valued matrix then $Y_{ij} = \sum_{r=1}^{N} \lambda_r \phi_{ir} \phi_{jr}$. Each (λ_r, ϕ_r) pair satisfies $\sum_{j=1}^{N} Y_{ij}\phi_{jr} = \lambda_r \phi_{ir}$ where $\{\lambda_r\}_{r=1}^{N}$ is a multiset of real numbers and $\{\phi_{ir}\}_{i,r=1}^{N}$ is an $N \times N$ orthogonal matrix with rth column denoted by ϕ_r.

Lemma B3 (Continuity): Let $f, g : [0,1]^2 \to \mathbb{R}$ be bounded symmetric measurable functions with positive eigenvalues $\{\lambda_r^+(f), \lambda_r^+(g)\}_{r=1}^{\infty}$ and negative eigenvalues

[*]Department of Economics, Northwestern University. E-mail: eric.auerbach@northwestern.edu.
[†]Department of Economics, Northwestern University. E-mail: yongcai2023@u.northwestern.edu.

$\{\lambda_r^-(f), \lambda_r^-(g)\}_{r=1}^{\infty}$ both ordered to be decreasing in absolute value. Suppose $\left(\int\int (f(u,v) - g(u,v))^2 \, dudv\right)^{1/2} \leq \epsilon$. Then $|\lambda_i^+(U) - \lambda_i^+(W)| \leq \epsilon$ and $|\lambda_i^-(U) - \lambda_i^-(W)| \leq \epsilon$. See Birman and Solomjak (2012), equation (19) following Theorem 8 in Chapter 9.2.

In Lemma 3 of Appendix Section A.1 in the main text, we use an implication of Lemma B3 and Theorem 368 of Hardy et al. (1952) (Lemma B5 below) that $\left(\sum_{r\in[R]} (\lambda_r(f) - \lambda_r(g))^2\right)^{1/2} \leq \sqrt{R} \left(\int\int (f(u,v) - g(u,v))^2 \, dudv\right)^{1/2}$ where $\{\lambda_r(f), \lambda_r(g)\}_{r\in[R]}$ are the R largest in absolute value eigenvalues of f and g ordered to be decreasing. This result is a crude analog of the Hoffman-Wielandt inequality for matrices (Lemma B6 below).

Lemma B4 (Birkhoff): For every $M \in \mathcal{D}_n^+$ there exists an $m \in \mathbb{N}$, $\alpha_1, ..., \alpha_m > 0$, and $P_1, ..., P_m \in \mathcal{P}_n$ such that $\sum_{t=1}^m \alpha_t = 1$ and $M_{ij} = \sum_{t=1}^m \alpha_t P_{ij,t}$. See Birkhoff (1946).

Lemma B5 (Hardy-Littlewood-Polya Theorem 368): For any $m \in \mathbb{N}$ and $g, h \in \mathbb{R}^m$ we have $\sum_{r=1}^m g_{(r)} h_{(m-r+1)} \leq \sum_{r=1}^m g_r h_r \leq \sum_{r=1}^m g_{(r)} h_{(r)}$ where $g_{(r)}$ is the rth order statistic of g. See Hardy et al. (1952), Section 10.2, Theorem 368.

Lemma B5 also holds for elements of $L^2([0,1])$, see Hardy et al. (1952), Section 10.13, Theorem 378. Specifically, for any $g, h \in L^2([0,1])$ we have $\int g^+(u) h^+(1-u) du \leq \int g(u) h(u) du \leq \int g^+(u) h^+(u) du$ where g^+ is the quantile function of g. This result is used in the second proof of Theorems 2.1 and 2.5 in Whitt (1976).

Lemma B6 (Hoffman-Wielandt): Let $\{\lambda_r(F)\}_{r\in[n]}$ and $\{\lambda_r(G)\}_{r\in[n]}$ be the eigenvalues of two $n \times n$ real symmetric matrices F and G, ordered to be decreasing. Then $\sum_{r=1}^n (\lambda_r(F) - \lambda_r(G))^2 \leq \sum_{i=1}^n \sum_{j=1}^n (F_{ij} - G_{ij})^2$. See Hoffman and Wielandt (1953).

C Examples

C.1 Examples of matrix rank invariant policy effects

C.1.1 Information diffusion

This example follows Banerjee et al. (2013); Cruz et al. (2017); Bramoullé and Genicot (2018). N agents are linked in a social network as described by an $N \times N$ symmetric binary adjacency matrix G. $G_{ij} = 1$ if agents i and j are linked and $G_{ij} = 0$ otherwise. Information about a new product or social program diffuses over the social network in T discrete time periods. In an initial period 0, one agent receives the relevant piece of information. For periods $t = 1, ..., T$, agents who received the information in period $t-1$, transmit it to their neighbors in period t. Specifically, if an agent receives the information M times from their neighbors in period $t-1$, the number of times they transmit the information to each of their neighbors in time period t is the sum of M independent Bernoulli(α) trials. The parameter α describes the probability that an agent will transmit information to their neighbors once they receive it.

The outcome of interest is the expected number of times agent j receives the information in the T time periods when agent i is initially informed in period 0. Proposition 1 of Bramoullé and Genicot (2018) implies that it is given by

$$Y_{ij} = \sum_{t=1}^{T} \alpha^t [G^t]_{ij}.$$

$[G^t]_{ij} = \sum_{s_1} \sum_{s_2} ... \sum_{s_{t-1}} G_{is_1} G_{s_1 s_2} ... G_{s_{t-1} j}$ is the ijth entry of the tth operator power of G.

Now consider an intervention that increases α, the probability of information transmission between agents. For example, the intervention may be a new advertisement campaign. Let $\alpha(s)$ be the transmission probability and $Y_{ij}(s)$ the resulting outcome matrix with and without the campaign as indexed by $s \in \{0, 1\}$. Then

$$Y_{ij}(s) = \sum_{r=1}^{n} \left(\sum_{t=1}^{T} \alpha(s)^t \lambda_r^t \right) \phi_{ir} \phi_{jr}$$

where (λ_r, ϕ_r) are the eigenvalue and eigenvector pairs of G. Under certain conditions on

T and $\alpha(1), \alpha(0)$, the treatment effect is matrix rank invariant in that $Y_{ij}](1) = g(Y_{ij}(0))$ where g is the lift of a nondecreasing function. For example, if $\alpha(1) > \alpha(0)$ and T is taken to infinity then $g(x) = \frac{\alpha(1)x}{\alpha(0)-(\alpha(1)-\alpha(0))x}$ which is increasing in x. More generally, if $\alpha(0) < \alpha(1) < \left(\frac{1}{T}\right)^{1/(T-2)}$ (for example, $\alpha(1), \alpha(0) < .5$ and $T > 3$) then g exists and is nondecreasing, although to our knowledge it does not have a tractable analytical representation.

C.1.2 Social interaction

This example follows Ballester et al. (2006); Calvó-Armengol et al. (2009). N agents are linked in a social network as described by an $N \times N$ symmetric binary adjacency matrix G. $G_{ij} = 1$ if agents i and j are linked and $G_{ij} = 0$ otherwise. Agents take K real-valued actions. The kth action of agent i is described by A_{ik}. It may describe, for example, how much agent i smokes or invests in a risky venture. The utility agent i receives from choosing action A_{ik} depends on the total amount of the action taken by their peers. Specifically,

$$U_i(A_k) = \eta_{ik} A_{ik} - \frac{1}{2} A_{ik}^2 + \beta \sum_{j=1}^{N} G_{ij} A_{jk} A_{ik}$$

where $\eta_{ik} \sim_{iid} (0, \sigma_k^2)$ is an idiosyncratic shock. The parameter β describes the size of the peer effect. That is, how much agents are influenced by their peers. Under the assumption that $I - \beta G$ is invertible, there exists a unique Nash equilibrium

$$A_k = (I - \beta G)^{-1} \eta_k$$

where $A_k = (A_{1k}, ..., A_{Nk})$.

The outcome of interest is the correlation of actions between agent pairs. It is given by

$$Y_{ij} = E\left[\sum_{k=1}^{K} A_{ik} A_{jk}\right] = \left[(I - \beta G)^{-1} \Sigma (I - \beta G)^{-1}\right]_{ij}$$

where $\Sigma_{ij} = E[\sum_{k=1}^{K} \eta_{ik}\eta_{jk}]$ is a diagonal matrix with $\Sigma_{ii} = \sum_{k=1}^{K} \sigma_k^2 := \sigma^2$.

Now consider an intervention that increases β, the peer effect size parameter. For example, the intervention may be a school program that better informs students about their peers' actions. Let $\beta(s)$ be the peer effects parameter and $Y_{ij}(s)$ the resulting outcome matrix with and without the program as indexed by $s \in \{0, 1\}$. Then

$$Y_{ij}(s) = \sum_{r=1}^{N} \left(\frac{\sigma^2}{(1 - \beta(s)\lambda_r)^2} \right) \phi_{ir}\phi_{jr}$$

where (λ_r, ϕ_r) are the eigenvalue and eigenvector pairs of G. If $\beta(1) > \beta(0) > 0$ then the treatment effect is matrix rank invariant in that $Y_{ij}(1) = g(Y_{ij}(0))$ where g is the matrix lift of a nondecreasing function. Specifically, $g(x) = \sigma^2 \left(1 - \frac{\beta(1)}{\beta(0)} \left(1 - \left(\frac{\sigma^2}{x}\right)\right)^{1/2}\right)^{-2}$ which is well defined and nondecreasing for x in the range of outcome matrices Y_0 that satisfy the condition that $I - \beta(0)G$ is invertible.

C.1.3 Link formation

This example follows Jochmans (2017); Graham (2017); Dzemski (2019). N agents are linked in a social network as described by an $N \times N$ symmetric binary adjacency matrix G. $G_{ij} = 1$ if i and j are linked and $G_{ij} = 0$ otherwise. The marginal transferable utility agents i and j receive from forming a link depends on their proximity in a K-dimensional social characteristic space. Specifically,

$$U_{ij}(G_{ij} = 1) - U_{ij}(G_{ij} = 0) = \alpha_i + \alpha_j - \beta \sum_{k=1}^{K} (x_{ik} - x_{jk})^2 + \eta_{ij}$$

where the fixed effect α_i describes agent i's degree heterogeneity or popularity, x_{ik} describes the kth social characteristic of agent i, and the idiosyncratic error η_{ij} is iid logistic. The parameter β describes the size of the homophily effect. That is, how much link formation is influenced by agent proximity in the social characteristic space.

The conditional probability that utility-maximizing agents i and j form a link is

$$E[G_{ij}] = \Lambda(\alpha_i + \alpha_j - \beta \sum_{k=1}^{K} (x_{ik} - x_{jk})^2).$$

where Λ is the standard logistic distribution function. We define the conditional logit

$$Y_{ij} = \Lambda^{-1}(E[G_{ij}]) = (\alpha_i + \beta \sum_k x_{ik}^2) + (\alpha_j + \beta \sum_k x_{jk}^2) + 2\beta \sum_k x_{ik}x_{jk}.$$

Our outcome of interest is the marginal transferable utility that a pair of agents receive from forming a connection. To simplify our exposition, we measure this outcome using the demeaned conditional logit

$$\tilde{Y}_{ij} = 2\beta \tilde{X}_{ij}$$

where $\tilde{Y}_{ij} = Y_{ij} - \frac{1}{N}\sum_{i=1}^N Y_{ij} - \frac{1}{N}\sum_{j=1}^N Y_{ij}$ and $\tilde{X}_{ij} = \sum_k x_{ik}x_{jk} - \frac{1}{N}\sum_{i,k=1}^N x_{ik}x_{jk} - \frac{1}{N}\sum_{j,k=1}^N x_{ik}x_{jk}$. It is straightforward to extend the logic of this example to the undemeaned conditional logit along the lines of Section D.2 below.

Now consider an intervention that increases β, the homophily size parameter. For example, the intervention may be a technology that decreases the costs of communication between locations. Let $\beta(s)$ be the homophily parameter and $\tilde{Y}_{ij}(s)$ the resulting outcome matrix with and without the communication technology as indexed by $s \in \{0, 1\}$. Then

$$\tilde{Y}_{ij}(s) = \sum_{r=1}^N (2\beta(s)\lambda_r)\, \phi_{ir}\phi_{jr}$$

where (λ_r, ϕ_r) are the eigenvalue and eigenvector pairs of $\tilde{X}$. If $\beta(1) > \beta(0) > 0$ then the treatment effect is matrix rank invariant in the sense of Definition 1 of Section 4.2.2 that $E[Y_{ij}](1) = g(E[Y_{ij}](0))$ where g is the matrix lift of a nondecreasing function. Specifically, $g(x) = \frac{\beta(1)}{\beta(0)}x$ which is increasing.

C.2 Examples of settings with spillovers, market externalities, etc.

C.2.1 Treatment spillovers

This example follows Bajari et al. (2021). They consider a buyer-seller experiment where pairs of buyers and sellers are randomly assigned to an information treatment. For example, the setting may be an online marketplace where a buyer-seller pair is treated if the platform explicitly recommends the seller's product to the buyer. The outcome of interest Y_{ij} is the size of the transaction between buyer i and seller j. Let $X_{ij} = 1$ if buyer i and seller j are assigned to treatment and $X_{ij} = 0$ otherwise.

Bajari et al. (2021) assume local interference (their Assumption 5.4). That is, the outcome Y_{ij} between buyer i and seller j depends on whether i and j are treated, the number of sellers l for which i and l are treated, and the number of buyers k for which k and j are treated. For example, buyer i may be more likely to buy from seller j if the platform recommends one of seller j's products, all else equal. Buyer i may be less likely to buy from seller j if the platform recommends products from seller j's competitors, all else equal.

Local interference suggests the model

$$Y_{ij} = f_{ij}\left(X_{ij}, \sum_l X_{il}, \sum_k X_{kj}\right).$$

Let $f_{ij}\left(x_b, x_s\right)$ be the expected outcome for agent pair ij when $\sum_l X_{il} = x_b$ and $\sum_k X_{kj} = x_s$ for some $x_b, x_s \in \mathbb{Z}_+$, i.e. $\sum_{t\in\{0,1\}} f_{ij}\left(t, x_b, x_s\right) P(X_{ij} = t)$. In this example, our parameter of interest is the distribution of treatment spillover effects

$$\frac{1}{NM} \sum_{i\in[N],j\in[M]} \mathbb{1}\{f_{ij}\left(x_b^1, x_s^1\right) - f_{ij}\left(x_b^0, x_s^0\right) \leq y\}.$$

In words, it is the fraction of buyer-seller pairs whose change in outcome, after altering the number of relevant treated agent pairs for i and j from (x_b^0, x_s^0) to (x_b^1, x_s^1) is less than y.

To identify the distribution of treatment spillover effects, we propose the following experiment. First randomly assign the treatment to pairs of buyers and sellers. Then form two groups. The first group collects all of the buyers that belong to exactly x_b^1 treated buyer-seller pairs and all of the sellers that belong to exactly x_s^1 treated buyer-seller pairs. The second group similarly collects all of the buyers and sellers that belong to x_b^0 and x_s^0 treated pairs. The next step is to use the matrix of outcomes associated with each group $Y_t := \{f_{ij}\left(X_{ij}, x_b^t, x_s^t\right)\}_{i,j\in \text{ group } t}$ to compute $\bar{Y}_t := \{f_{ij}\left(x_b^t, x_s^t\right)\}_{i,j\in \text{ group } t}$. Finally, after symmetrization as in Section 5.1, the distribution of spillover effects can be characterized exactly as in Section 4 of the main text.

C.2.2 Market externalities

Consider a market economy with N agents and L goods. For a fixed price $p \in \mathbb{R}^{L-1}$, agent i's demand for the lth good is given by the function $Q_{il}(p)$ with $Q_i(p) = \{Q_{i1}(p), ..., Q_{iL}(p)\} \in \mathbb{R}^L$. Q_{il} may be negative in which case i is a supplier of good l. An equilibrium market price $p^*(0)$ is assumed to satisfy the market clearing condition $\sum_{i=1}^N Q_i^*(0) = 0$ where $Q_i^*(0) = Q_i(p^*(0))$ is agent i's equilibrium demand. Absent any market intervention, the equilibrium price and quantity $(p^*(0), Q_1^*(0), ..., Q_N^*(0))$ is realized.

We are interested in understanding the impact of a market intervention such as a price floor on the equilibrium demand matrix between agents and goods. An equilibrium market price $p^*(1)$ is assumed to satisfy the market clearing condition $\sum_{i=1}^N Q_i^*(1) = 0$ and restriction $p^*(1) \geq c$ where $Q_i^*(1) = Q_i(p^*(1))$ is agent i's equilibrium demand under the price floor and $c \in \mathbb{R}^{L-1}$ is chosen by the policy maker. Under the price floor intervention, the equilibrium price and quantity $(p^*(1), Q_1^*(1), ..., Q_N^*(1))$ is realized (if no equilibrium exists then we assume that the policy maker allocates the goods in some other deterministic way).

Our interest is in the distribution of treatment effects

$$\frac{1}{NL} \sum_{i \in [N], l \in [L]} \mathbb{1}\{Q_{il}^*(1) - Q_{il}^*(0) \leq y\}.$$

In words, it is the fraction of agent and good pairs whose difference in equilibrium demand with and without the price ceiling is less than y. There are market externalities in this example because while price floor may only nominally restrict one agent or item, the equilibrium condition implies that implementing the policy may result in changes in the equilibrium demand matrix for any agent and item.

To identify the distribution of treatment effects with market externalities, we suppose that the researcher is given data from the following natural experiment. They observe a matrix of equilibrium demand for a population of agents and goods from a region without the price floor. They observe another matrix of equilibrium demand for a different population of agents and goods from a region with the price floor. The two regions may not have any agents or goods in common, but they are assumed to be comparable in the sense of Assumption 1 in Section 3.3 of the main text. For example, the regions may have similar economic activity but be under different political jurisdictions. After symmetrization as in Section 5.1, the two outcome matrices can be used to characterize the distribution of equilibrium treatment effects exactly as in Section 4 of the main text.

C.3 Example where randomization implies Assumption 1

We specify a general large sample strategic network formation model under which randomizing agents to treatment groups implies Assumption 1 from Section 3.3 of the main text. Related models are considered, for example, by Leung (2015); Menzel (2015); Ridder and Sheng (2015); Badev (2017); Mele (2017); Thirkettle (2019). To simplify our exposition and minimize notation we assume that network formation is deterministic, but the logic of our model is straightforward to apply to a stochastic model as well.

In words, the model we consider is one where agents make strategic decisions to form connections and the observed network is in an equilibrium where no pair of agents has an incentive to alter their relationship status. The equilibrium connections are determined by exactly two things: the exogenous characteristics of the agents and the policy implemented by the researcher. By randomly assigning the agents to the treatment groups, the researcher ensures that that their characteristics are similar to the population of interest. As a result, the equilibrium connections between the agents in the treatment groups necessarily reveal how the agents in the population of interest would be connected under the same policy.

Formally, we specify a model of network formation that is defined on any group of agents indexed by $[0, 1]$. The size of the group is infinite. Each agent in the group is endowed with a vector of characteristics described by the measurable function $X(\cdot) : [0, 1] \to \mathbb{R}^k$ for some positive integer k. The agents' characteristics are exogenous in that they do not vary with the policy implemented or connections formed between agents in the group. Some or all of the characteristics may be unobserved by the researcher.

The agents in the group interact and form binary network connections. The marginal utility U that an agent indexed by $u \in [0, 1]$ gets from forming a connection with an agent indexed by $v \in [0, 1]$ depends on the policy implemented in the group $t \in \{0, 1\}$, the characteristics of *any* of the agents in the group $X(\cdot)$, and *any* of the other connections between pairs of agents in the group $Y_t(\cdot,\cdot) : [0,1]^2 \to \mathbb{R}$. That is,

$$U(u,v) = f_t((u,v), X(\cdot), Y_t(\cdot,\cdot)).$$

In words, if agents u and v are not connected in the network ($Y_t(u,v) = 0$) then forming a connection gives $f_t((u,v), X(\cdot), Y_t(\cdot,\cdot))$ additional utility to agent u. If agents u and v are connected ($Y_t(u,v) = 1$) then destroying that connection gives $-f_t((u,v), X(\cdot), Y_t(\cdot,\cdot))$ additional utility to agent u.

Our model is agnostic to how the agents actually form their connections in the network. Instead, we only suppose that whatever the network formation process is, it results in connections that satisfy a particular equilibrium condition: two agents are connected if and only if both agents receive a positive marginal utility from the connection. That is, $Y_t(\cdot,\cdot)$ is assumed to be measurable and satisfy the rule

$$Y_t(u,v) = \mathbb{1}\{f_t((u,v), X(\cdot), Y_t(\cdot,\cdot)) > 0\}\mathbb{1}\{f_t((v,u), X(\cdot), Y_t(\cdot,\cdot)) > 0\} \tag{1}$$

for every $u, v \in [0,1]$. Under this rule, the network is said to be pairwise stable, see originally Jackson and Wolinsky (1996).[1]

We make two main assumptions about the equilibrium network connections. These assumptions are strong but typical of the literature on strategic network formation. The first main assumption is that either (1) has a unique equilibrium or that there is a deterministic equilibrium selection process. This assumption implies that the equilibrium connections can be represented by the reduced form equation

$$Y_t(u,v) = g_t((u,v), X(\cdot)). \tag{2}$$

If this condition does not hold, then Assumption 1 may be violated even when agents are randomized to the treatment groups because the equilibrium connections chosen by the treatment groups may not be representative of what the population of interest would choose

[1] The choice of pairwise stability as an equilibrium concept is not necessary for this section. Any other equilibrium concept would work, subject to the conditions outlined below.

under the same policy. A researcher could potentially relax this unique equilibrium assumption by finding all of the possible equilibria, characterizing the parameters of interest for each possibility separately, and then taking the union of the identified sets. This could be done by having a large number of treatment groups (so that every possible equilibrium is eventually observed in the experiment), or by choosing a specific parametric form of the utility function and solving for all of the equilibria analytically.

The second main assumption is a standard continuity condition: g_t is continuous in $X(\cdot)$ for almost every agent pair. That is, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sup_{x \in \mathbb{R}^k} \left| \int \mathbb{1}\{X(u) \leq x\} du - \int \mathbb{1}\{X^*(u) \leq x\} du \right| \leq \delta$ implies that $\int \int |g_t((u,v), X(\cdot)) - g_t((u,v), X^*(\cdot))| \, du dv \leq \epsilon$. In words, this second assumption says that the existence of a connection between agents u and v in equilibrium is robust to small perturbations in the distribution of covariates in the group. If this condition does not hold, then Assumption 1 may be violated even when agents are randomized to treatment groups because even though randomization may ensure that the distribution of characteristics in the treatment groups are similar to those in the population of interest, small discrepancies between the two groups may still result in large differences in the equilibrium connections between agents.

Under our two main assumptions, the equilibrium network connections are determined by only two factors: the distribution of the characteristics of the agents in the group and the policy implemented by the researcher. Randomization is a way for the researcher to ensure that the distribution of the characteristics in the treatment groups matches the population of interest. Specifically, suppose that the population of interest has a distribution of characteristics given by $X^*(\cdot)$. Then the first main assumption implies that the equilibrium network connections between agents in the population under policy t is $Y_t^*(u,v) = g_t((u,v), X^*(\cdot))$. In the experiment, the researcher draws a representative sample of N agents from the population and randomly assigns them to treatment groups 0 and 1 so that the distribution of characteristics in treatment group t is given by $X_t(\cdot)$. The policy is implemented in group 1 but not group 0. Agents interact and form equilibrium network connections given by the function Y_t. Then by the Glivenko-Cantelli Theorem, $\sup_{x \in \mathbb{R}^k} \left| \int \mathbb{1}\{X_t(u) \leq x\} du - \int \mathbb{1}\{X^*(u) \leq x\} du \right| = o_p(1)$ for $t \in \{0, 1\}$. It follows from the second main assumption that $Y_1(u,v) = g_1((u,v), X^*(\cdot)) = Y_1^*(u,v) + o_p(1)$ and

$Y_0(u,v) = g_0((u,v), X^*(\cdot)) = Y_0^*(u,v) + o_p(1)$ for almost every u, v. As a result, in large samples, Y_t and Y_t^* must have the same homomorphism densities as in Section 3.3 of the main text. Assumption 1 follows.

D Extensions

D.1 Asymmetric outcome matrices

Asymmetric matrices or matrices indexed by two different populations are handled in the following way. A population of workers and firms are randomized (or as good as randomized) to a treatment ($t = 1$) and control ($t = 0$) group. A policy is implemented in only one of the groups. For example, the groups may correspond to economic regions where one region is exposed to a trade shock (the policy) and the other is not (the status quo).

Potential outcomes are defined for each worker and firm pair. We index the workers with $[0,1]$ and firms with $[2,3]$. These index sets are arbitrary and only used to differentiate between the two types of agents. The potential outcomes are represented by $(Y_0^*, Y_1^*) : S \to \mathbb{R}^2$ where $S = [0,1] \times [2,3]$. For example, $Y_1^*(u,v)$ may describe the potential wage that a worker with index u would earn at a firm with index v when exposed to the trade shock. Following Assumption 1, we assume that the researcher observes Y_1 and Y_0 where $Y_t(\varphi_t(u), \psi_t(v)) = Y_t^*(\tilde{\varphi}_t(u), \tilde{\psi}_t(v))$ for unknown $\phi_t, \tilde{\varphi}_t, \psi_t, \tilde{\psi}_t \in \mathcal{M}$. The DPO is $F(y_1, y_0) = \int\int \prod_{t\in\{0,1\}} \mathbb{1}\{Y_t(\phi_t(u), \psi_t(v)) \leq y_t\} du dv$ and the DTE is $\Delta(y) = \int\int \mathbb{1}\{Y_1(\phi_1(u), \psi_1(v)) - Y_0(\phi_0(u), \psi_0(v)) \leq y\} du dv$.

We symmetrize the potential outcome matrices along the lines of Auerbach (2022b). Let $S^2 = ([0,1] \cup [2,3]) \times ([0,1] \cup [2,3])$ and define $(Y_0^\dagger, Y_1^\dagger) : S^2 \to \mathbb{R}^2$ so that

$$Y_t^\dagger(u,v) := \begin{cases} Y_t(u,v) & \text{if } (u,v) \in [0,1] \times [2,3] \\ Y_t(v,u) & \text{if } (u,v) \in [2,3] \times [0,1] \\ \infty & \text{otherwise} \end{cases}$$

and $\varphi_t(u) := \phi_t(u)\mathbb{1}\{u \in [0,1]\} + \psi_t(u)\mathbb{1}\{u \in [2,3]\}$ is measure preserving. Then the DPO is equal to $\frac{1}{2}\int\int\prod_{t\in\{0,1\}}\mathbb{1}\{Y_t^\dagger(\varphi_t(u),\varphi_t(v))\leq y_t\}dudv$. Since $Y^\dagger$ is symmetric and defined on one population (the population of workers and firms), the logic of Sections 3-4 can be applied to bound the DPO and DTE. One can similarly define the STE using the eigenvalues of $Y_t^\dagger$, although in that case we substitute 0 for ∞ in the definition of $Y_t^\dagger$.

D.2 Row and column heteroskedasticity

While the bounds from Section 4 are valid for any symmetric outcome matrices, they may be uninformative when there is nontrivial heterogeneity in the row and column variances. In such cases, we propose an adjustment building on Section 5 of Finke et al. (1987). Let $\mathbb{1}\{Y_t^*(u,v)\leq y_t\}=\alpha_t(u)+\alpha_t(v)+\epsilon_t(u,v)$ where $\int\epsilon_t(s,v)ds=\int\epsilon_t(u,s)ds=0$ for every $u,v\in[0,1]$.

D.2.1 Bounds on the DPO and DTE

The DPO becomes

$$F(y_1,y_0)=\int\int\prod_{t\in\{0,1\}}\left(\alpha_t(\varphi_t(u))+\alpha_t(\varphi_t(v))+\epsilon_t(\varphi_t(u),\varphi_t(v))\right)dudv$$
$$=\int\int\prod_{t\in\{0,1\}}\left(\alpha_t(\varphi_t(u))+\alpha_t(\varphi_t(v))\right)dudv+\int\int\prod_{t\in\{0,1\}}\epsilon_t(\varphi_t(u),\varphi_t(v))dudv.$$

We bound the two summands separately. Specifically, the upper bound is

$$F(y_1,y_0)\leq\max_{\varphi_1,\varphi_0\in\mathcal{M}}\left[\int\int\prod_{t\in\{0,1\}}\left(\alpha_t(\varphi_t(u))+\alpha_t(\varphi_t(v))\right)dudv+\int\int\prod_{t\in\{0,1\}}\epsilon_t(\varphi_t(u),\varphi_t(v))dudv\right]$$
$$\leq\max_{\varphi_1,\varphi_0\in\mathcal{M}}\left[\int\int\prod_{t\in\{0,1\}}\left(\alpha_t(\varphi_t(u))+\alpha_t(\varphi_t(v))\right)dudv\right]+\max_{\varphi_1,\varphi_0\in\mathcal{M}}\left[\int\int\prod_{t\in\{0,1\}}\epsilon_t(\varphi_t(u),\varphi_t(v))dudv\right].$$

The first summand is bounded from above by

$$2\max_{\varphi_1,\varphi_0\in\mathcal{M}}\left[\int\alpha_1(\varphi_1(u))\alpha_0(\varphi_0(u))du\right]+2\alpha_1\alpha_0\leq2\int\alpha_1^+(u)\alpha_0^+(u)du+2\alpha_1\alpha_0$$

where $\alpha_t=\int\alpha_t(u)du$ and α_t^+ is the quantile function of α_t. See Theorem 378 of Hardy et al. (1952) or the second proof of Theorems 2.1 and 2.5 of Whitt (1976) for details. Following Proposition 2 of the main text, the second summand is bounded from above by

$$\min\left(\sum_r \lambda_{r1}^2, \sum_r \lambda_{r0}^2, \sum_r \lambda_{r1}\lambda_{r0}\right)$$

where λ_{rt} refers to the rth eigenvalue of ϵ_t and the sums are defined as in Section 4.1.2 of the main text. Together, the bounds imply that

$$F(y_1, y_0) \leq 2\int \alpha_1^+(u)\alpha_0^+(u)du + 2\alpha_1\alpha_0 + \min\left(\sum_r \lambda_{r1}^2, \sum_r \lambda_{r0}^2, \sum_r \lambda_{r1}\lambda_{r0}\right).$$

By the same logic, the lower bound on the DPO is

$$F(y_1, y_0) \geq 2\int \alpha_1^+(u)\alpha_0^+(1-u)du + 2\alpha_1\alpha_0 + \max\left(\sum_r \left(\lambda_{r1}^2 + \lambda_{r0}^2\right) - 1, \sum_r \lambda_{r1}\lambda_{s(r)0}, 0\right).$$

Bounds on the DTE can be constructed from those on the DPO following the logic of Proposition 3 in Section 4 of the main text.

D.2.2 Spectral treatment effects

We suppose the rank invariance assumptions $\alpha_1(\varphi_1(u)) = g_\alpha(\alpha_0(\varphi_0(u))$ and $\epsilon_1(\varphi_1(u), \varphi_1(v)) = g_\epsilon(\epsilon_0(\varphi_0(u), \varphi_0(v))$ for every $u, v \in [0,1]$ where g_α is a nondecreasing function and g_ϵ is the matrix lift of a nondecreasing function as in Definition 1 of Section 4.2.2. Define the spectral treatment effect with row and column heterogeneity to be

$$STE(u, v; \phi) = \left(\alpha_1^+(u) - \alpha_0^+(u)\right) + \left(\alpha_1^+(v) - \alpha_0^+(v)\right) + \sum_r (\sigma_{r1} - \sigma_{r0})\phi_r(u)\phi_r(v)$$

where $\{\phi_r\}$ is any orthogonal basis in $L^2([0,1])$ and $\{\sigma_{rt}\}$ are the eigenvalues of ϵ_t. Similarly define $STT(u, v) = STE(u, v; \phi_1)$ and $STU(u, v) = STE(u, v; \phi_0)$ where ϕ_1 and ϕ_0 refer to the eigenfunctions of ϵ_1 and ϵ_0 respectively.

Then by the logic of Proposition 5 in Section 4 of the main text

$$\begin{aligned} Y_1^*(u,v) - Y_0^*(u,v) &= (\alpha_1(\varphi_1(u)) - \alpha_0(\varphi_0(u))) + (\alpha_1(\varphi_1(v)) - \alpha_0(\varphi_0(v))) \\ &\quad + (\epsilon_1(\varphi_1(u), \varphi_1(v)) - \epsilon_0(\varphi_0(u), \varphi_0(v))) \\ &= (g_\alpha(\alpha_0(\varphi_0(u))) - \alpha_0(\varphi_0(u))) + (g_\alpha(\alpha_0(\varphi_0(v))) - \alpha_0(\varphi_0(v)) \\ &\quad + (g_\epsilon(\epsilon_0(\varphi_0(u), \varphi_0(v))) - \epsilon_0(\varphi_0(u), \varphi_0(v))) \\ &= (g_\alpha(\alpha_0(\varphi_0(u))) - \alpha_0(\varphi_0(u))) + (g_\alpha(\alpha_0(\varphi_0(v))) - \alpha_0(\varphi_0(v)) \\ &\quad + \sum_r (g_\epsilon(\sigma_{r0}) - \sigma_{r0})\phi_{r0}(\varphi_0(u))\phi_{r0}(\varphi_0(v)) \\ &= STE(\varphi_0(u), \varphi_0(v); \phi_0)) \end{aligned}$$

Since ε_1 and ε_0 are matrix rank invariant, they have the same eigenfunctions (see the proof of Proposition 5 in Section A.5), and so $STE(\varphi_0(u), \varphi_0(v); \phi_0)) = STE(\varphi_0(u), \varphi_0(v); \phi_1))$. It follows that under the rank invariance assumption $Y_1^* - Y_0^*$, STT, and STU all have the same marginal distribution function.

D.3 Randomization inference

The focus of our paper is on identification, but for completeness we give two examples showing how one can conduct randomization based inference about the disruptive impact of a policy. We focus on the global point null of no treatment effect, take the populations to be finite and the potential outcomes to be fixed, and do not explicitly consider network interference. One can, however, similarly consider infinite populations, test other hypotheses, or invert the tests to construct point estimates and confidence intervals in the sense of Hodges and Lehmann (2012); Rosenbaum (2002), see also Athey et al. (2018); Basse et al. (2019a;b).

D.3.1 Double randomization with uncensored outcomes

This example is based on the conjunctive simple multiple randomization design of Bajari et al. (2021). A group of B buyers and S sellers are independently randomized to one of two groups. The probability that any buyer or seller is assigned to group 1 is $\pi \in (0, 1)$. Every buyer-seller pair where both the buyer and the seller of that pair are assigned to group 1 is given an information treatment.

Let $Y_{ij,st}$ be the potential transaction between buyer i and seller j in the event that i is assigned to group $s \in \{0, 1\}$ and j is assigned to group $t \in \{0, 1\}$. $\tilde{Y}_{ij}$ is the observed transaction for buyer i and seller j under their realized group assignments. We call this example uncensored because the researcher observes some transaction value for every pair of agents in the experiment (that value may be 0). To simplify arguments, we assume that the potential transactions for a buyer-seller pair do not depend on exactly which other buyers or sellers are assigned to groups 1 and 2. This corresponds to the authors' Assumption 5.1 and may be the case when the number of buyers and sellers assigned to each group is large as in Section C.3.

The null hypothesis is that the information treatment is not disruptive. That is, it has no effect on the potential transactions between buyers and sellers in the marketplace. That is,

$$H_0 : Y_{ij,st} = Y_{ij,s't'} \text{ for every } i \in [B], j \in [S], s, s', t, t' \in \{0, 1\}.$$

For a test statistic, we propose the difference in the eigenvalues of the outcome matrices associated with each treatment

$$T = \max_{s,s',t,t' \in \{1,2\}} \sup_{y \in \mathbb{R}} \sum_r \left(\lambda_{r,st}(y) - \lambda_{r,s't'}(y)\right)^2 .$$

where $\lambda_{r,st}(y)$ is the rth eigenvalue of $\{\mathbb{1}\{\tilde{Y}^{\dagger}_{ij} \leq y\}/N_{st}\}_{i,j \in B(s) \cup S(t)}$, $B(s) = \{i \in B : i \text{ is assigned to group } s\}$ and $S(t) = \{j \in S : i \text{ is assigned to group } t\}$, and $N_{st} = |B(s)| + |S(t)|$. The logic of this test statistic follows Proposition 4 of the main text. It is a conservative measure of the total amount of disruption caused by a change in policy.

For a reference distribution, we rerandomize the individual treatment assignments. For any positive integer A and $a \in [A]$, let $\rho^B_{i,a}$ and $\rho^S_{j,a}$ be a collection of independent Bernoulli(π) random variables, $B^a(s) = \{i \in B : \rho^B_{i,a} = s\}$ and $S^a(t) = \{j \in S : \rho^S_{j,a} = t\}$ be the set of buyers and sellers rerandomized to group s and t respectively,

$$\tilde{Y}^a_{st} = \{\tilde{Y}_{ij}\}_{i \in B^a(s), j \in S^a(t)}$$

and

$$T^a = \max_{s,s',t,t' \in \{1,2\}} \sup_{y \in \mathbb{R}} \sum_r \left(\lambda^a_{r,st}(y) - \lambda^a_{r,s't'}(y) \right)^2$$

where $\lambda^a_{r,st}(y)$ is the rth eigenvalue of $\mathbb{1}\{\tilde{Y}^{a\dagger}_{st} \leq y\}/\tilde{N}^a_{st}$ and $\tilde{N}^a_{st} = |B^a(s)| + |S^a(t)|$.

By Lehmann and Romano (2006) Theorem 15.2.1, the test that rejects H_0 whenever

$$(A+1)^{-1} \left(1 + \sum_{a \in [A]} \mathbb{1}\{T^a \geq T\} \right) \leq \alpha$$

is level α. It is powered to detect deviations in the eigenvalues of the transaction matrices associated with each treatment. That such deviations detect a large class of socially disruptive policy effects follows Propositions 3-4 in the main text.

D.3.2 Double randomization with censored outcomes

This example is based on Comola and Prina (2021). A collection of households are randomly chosen to participate in a savings program. Each household is assigned to participate independently with probability $\pi \in (0, 1)$.

Let $Y_{ij,t}$ be the potential risk sharing link for households i and j when both ($t = 1$) or neither ($t = 0$) participate in the program. $\tilde{Y}_{ij,t}$ is the observed risk sharing link for households i and j that were actually assigned to treatment t. $\tilde{N}_t$ is the number of households actually assigned to treatment t. We call this example censored because in this example the researcher only observes the potential risk sharing links for pairs of agents assigned to the same treatment. To simplify our exposition, we assume that the number of participants is small relative to the number of non-participants (i.e. $\pi \approx 0$).

The null hypothesis is that participation in the savings program has no effect on the potential risk sharing links between pairs of households. That is,

$$H_0 : Y_{ij,t} = Y_{ij,t'} \text{ for every } t, t' \in \{0, 1\}.$$

For a test statistic, we propose the difference in the eigenvalues of the outcome matrices associated with the treated and untreated household pairs

$$T = \sup_{y \in \mathbb{R}} \sum_r \left(\lambda_{r,1}(y) - \lambda_{r,0}(y)\right)^2$$

where $\lambda_{r,t}(y)$ is the rth eigenvalue of $\mathbb{1}\{\tilde{Y}_t \leq y\}/\tilde{N}_t$. As in the previous example, the logic of this test statistic follows from Proposition 4 of the main text.

For a reference distribution, we rerandomize the individual treatment assignments. For any positive integer A and $a \in [A]$, let $\rho_{i,a}$ be a collection of independent Bernoulli(π) random variables,

$$\tilde{Y}_1^a = \{Y_{ij,0}\}_{\rho_{i,a}=1, \rho_{j,a}=1},$$

and

$$T^a = \sup_{y \in \mathbb{R}} \sum_r \left(\lambda_{r,1}^a(y) - \lambda_{r,0}(y)\right)^2$$

where $\lambda_{r,1}^a$ is the rth eigenvalue of $\mathbb{1}\{\tilde{Y}_1^a \leq y\}/\tilde{N}_1^a$ and $\tilde{N}_1$ is the number of households rerandomized to treatment 1.

By Lehmann and Romano (2006) Theorem 15.2.1, the test that rejects H_0 whenever

$$(A+1)^{-1}\left(1 + \sum_{a \in [A]} \mathbb{1}\{T^a \geq T\}\right) \leq \alpha$$

is level α. It is powered to detect deviations in the eigenvalues of the networks associated with each treatment. That such deviations detect a large class of socially disruptive policy effects follows Proposition 2 in the main text.

D.4 Large sample estimation and inference

The focus of our paper is on identification, but for completeness we also show how one can estimate and conduct inference about the bounds on the DPO, DTE, and the distribution of STE using sampled, mismeasured, or missing data.

The data is assumed to be drawn from an infinite population. Our estimators use kernel density smoothing along the lines of Horowitz (1992). Alternative strategies may lead to more accurate inferences in practice, but we leave their study to future work.

D.4.1 Assumptions about the data generating process

We consider the setting of Section 3 in the main text, except the researcher does not observe the network connections in the treatment groups $Y_t : [0,1]^2 \to \mathbb{R}$ for $t \in \{0,1\}$. Instead, they have a stochastic approximation $\hat{Y}_t : [0,1]^2 \to \mathbb{R}$ whose accuracy depends on a sample size. We give more information about the assumed relationship between $\hat{Y}_t$ and Y_t below.

For example, the researcher may observe the $N \times N$ matrix M_t with ijth entry $M_{ij,t} = (Y_t(w_{i,t}, w_{j,t}) + \epsilon_{ij,t})\,\eta_{ij,t}$. N agents are sampled from the infinite population as described by $w_{i,t} \sim_{iid} F_w$. The connections between pairs of agents are observed with measurement error as described by $\epsilon_{ij,t} \sim_{iid} F_\epsilon$. Some outcomes are missing at random as described by $\eta_{ij,t} \sim_{iid} Bernoulli(p_t)$ with $p_t \in (0,1)$. To construct $\hat{Y}_t$, the researcher first estimates the entries of $Y_t(w_{i,t}, w_{j,t})$ conditional on $\{w_{i,t}\}_{i \in [N]}$. This may be done by local averaging, k-means clustering, linear regression, PCA, spectral thresholding, etc. See for instance Bai et al. (2008); Bonhomme and Manresa (2015); Chatterjee (2015); Stock and Watson (2016); Jochmans and Weidner (2019); Graham (2020). $\hat{Y}_t$ is then the function embedding (see Appendix Section A.1.1) of this matrix of estimates, potentially weighted by the inverse density of $w_{i,t}$.

To demonstrate consistency of our estimators (specified below), our main assumption is that $\hat{Y}_t$ is consistent in mean squared error. That is,

Assumption D1:

$$MSE(\hat{Y}) := \max_{t \in \{0,1\}} \int \int \left(\hat{Y}_t(u,v) - Y_t(u,v) \right)^2 du dv \to_p 0 \text{ as } N \to \infty.$$

Nearly all of the methods proposed in the literature are designed to satisfy this property under certain regularity conditions. We show that under Assumption 1 and additional regularity conditions, the mean squared error of our estimators vanishes with $MSE(\hat{Y}_t)$. It follows that our estimators are consistent under Assumption D1.

To construct a distribution for large sample inference, our main assumption is that Y_t is determined by a linear model. Specifically,

Assumption D2: For $t \in \{0,1\}$, $Y_t(u,v) = X_t(u,v)\beta_t$ and $\hat{Y}_t(u,v) = X_t(u,v)\hat{\beta}_t$ with $\beta_t, \hat{\beta}_t \in \mathbb{R}^K$ for some $K \in \mathbb{N}$, $\left(\hat{\beta}_t - \beta_t\right) \to_d \mathcal{N}(0, V_t)$ as $N \to \infty$ for some covariates $X_t(u,v)$, V_t can be consistently estimated, and $\hat{\beta}_1$ and $\hat{\beta}_0$ have independent entries.

Linear models are common in network economics, see for instance Bonhomme and Manresa (2015); Jochmans and Weidner (2019); Auerbach (2022a). We suspect that it is straightforward to extend the arguments of this section to nonlinear or nonparametric models in the usual way, but we do not have space in this online appendix to demonstrate additional results.

D.4.2 Additional regularity conditions

We rely on the following regularity conditions. They are analogous to Assumptions K1, K2, 6, and 9 in Horowitz (1992), but modified to fit our setting.

Assumption D3:

i $K : \mathbb{R} \to \mathbb{R}$ is everywhere twice differentiable with $|K|$, $|K'|$, and $|K''|$ uniformly bounded, $\lim_{u\to\infty} K(u) = 0$ and $\lim_{u\to-\infty} K(u) = 1$. $\int [K'(u)]^4\, du$, $\int [K''(u)]^2\, du$, and $\int [u^2 K''(u)]^4\, du$ are finite. For some $P \in \mathbb{N}$, $P \geq 2$ and $p \in [P]$ $\int |u^p K'(u)| du$ is also finite with $\int u^p K'(u) du = 0$.

ii $h(N)$ is a bandwidth sequence such that $h \to 0$, $h^{p-P-1} \int_{|hu|>\eta} |u^p K'(u)| du \to 0$, $h^{-1} \int_{|hu|>\eta} |K''(u)| du \to 0$, and $h^{-P} MSE(\hat{Y}) \to \infty$ as $N \to \infty$ for any $\eta > 0$ and $p \in [P]$.

iii The marginal distribution function of the STE parameter, $F_{STE}(y;\phi) = \int\int \mathbb{1}\{STE(u,v;\phi) \leq y\} du dv$, is everywhere smooth with uniformly bounded derivatives for a fixed orthogonal basis $\{\phi_r\}$ of $L_2([0,1])$.

Assumption D4:

i $K:\mathbb{R}\to\mathbb{R}$ is everywhere twice differentiable with $|K|$, $|K'|$, and $|K''|$ uniformly bounded, $\lim_{u\to\infty}K(u)=0$, $K(0)=1$, and $\lim_{u\to-\infty}K(u)=1$. $\int[K'(u)]^4\,du$, $\int[K''(u)]^2\,du$, and $\int[u^2K''(u)]^4\,du$ are finite. For some $P\in\mathbb{N}$, $P\geq 2$ and $p\in[P]$, $\int|u^pK'(u)|du$ is also finite with $\int_{u\leq 0}u^pK'(u)du=\int u^pK(u)K'(u)du=0$ for $p\in[P]$.

ii $h(N)$ is a bandwidth sequence such that $h\to 0$, $h^{p-P-1}\int_{|hu|>\eta}|u^pK'(u)|du\to 0$, $h^{-1}\int_{|hu|>\eta}|K''(u)|du\to 0$, and $h^{-P}MSE(\hat{Y})^2\to\infty$ as $N\to\infty$ for any $\eta>0$ and $p\in[P]$.

iii The marginal distribution function of the STE parameter, $F_{STE}(y;\phi)=\int\int\mathbb{1}\{STE(u,v;\phi)\leq y\}dudv$, is everywhere smooth with uniformly bounded derivatives for a fixed orthogonal basis $\{\phi_r\}$ of $L_2([0,1])$.

A consequence of Assumption D4(i) is that $2\int K(u)K'(u)du=-1$ and $\int_{u\leq 0}K'(u)=0$.

D.4.3 Consistent estimation of the bounds on the DPO and DTE

Let $\{\lambda_{rt}\}_{r\in[R]}$ and $\{\hat{\lambda}_{rt}\}_{r\in[R]}$ denote the R largest eigenvalues in absolute value of the functions $\mathbb{1}\{Y_t(\cdot,\cdot)\leq y_t\}$ and $K\left(\left(\hat{Y}_t(\cdot,\cdot)-y_t\right)/h\right)$, ordered to be decreasing. To estimate the bounds on the DPO and DTE, we propose $\sum_{r\in\mathbb{N}}\hat{\lambda}_{rt}\hat{\lambda}_{rt'}$ to estimate $\sum_{r\in\mathbb{N}}\lambda_{rt}\lambda_{rt'}$ for $t,t'\in\{0,1\}$. We show that

Proposition D1: Under Assumptions D1 and D4

$$\sup_{y_t,y_{t'}\in\mathbb{R}}\left|\sum_{r\in\mathbb{N}}\hat{\lambda}_{rt}\hat{\lambda}_{rt'}-\sum_{r\in\mathbb{N}}\lambda_{rt}\lambda_{rt'}\right|=O_p\left(MSE\left(\hat{Y}\right)\right).$$

Proof of Proposition D1: Write

$$\left|\sum_{r\in\mathbb{N}}\hat{\lambda}_{rt}\hat{\lambda}_{rt'}-\sum_{r\in\mathbb{N}}\lambda_{rt}\lambda_{rt'}\right|\leq\left|\sum_{r\in\mathbb{N}}\left(\hat{\lambda}_{rt}-\lambda_{rt}\right)\lambda_{rt'}\right|+\left|\sum_{r\in\mathbb{N}}\left(\hat{\lambda}_{rt'}-\lambda_{rt'}\right)\lambda_{rt}\right|+r_N$$

where $r_N=\left|\sum_{r\in\mathbb{N}}\left(\hat{\lambda}_{rt}-\lambda_{rt}\right)\left(\hat{\lambda}_{rt'}-\lambda_{rt'}\right)\right|$ is asymptotically negligible relative to the first two terms. The first two summands are bounded

$$\left|\sum_r \left(\hat{\lambda}_{rt} - \lambda_{rt}\right)\lambda_{rt'}\right| \leq \left[\sum_r \left(\hat{\lambda}_{rt} - \lambda_{rt}\right)^2\right]^{1/2} F_{t'}(y_{t'})^{1/2}$$

by Cauchy-Schwarz since $\left(\sum_r \lambda_{rt}^2\right)^{1/2} = F_t(y_t)^{1/2}$. The term $\left[\sum_r \left(\hat{\lambda}_{rt} - \lambda_{rt}\right)^2\right]^{1/2}$ is further bounded

$$\begin{aligned}
\left[\sum_r \left(\hat{\lambda}_{rt} - \lambda_{rt}\right)^2\right]^{1/2} &\leq \left[\int\int \left(K\left(\frac{\hat{Y}_t(u,v) - y_t}{h}\right) - \mathbb{1}\{Y_t(u,v) \leq y_t\}\right)^2 dudv\right]^{1/2} \\
&\leq \left[\int\int \left(K\left(\frac{\hat{Y}_t(u,v) - y_t}{h}\right) - K\left(\frac{Y_t(u,v)}{h}\right)\right)^2 dudv\right]^{1/2} \\
&\quad + \left[\int\int \left(K\left(\frac{Y_t(u,v) - y_t}{h}\right) - \mathbb{1}\{Y_t(u,v) \leq y_t\}\right)^2 dudv\right]^{1/2}.
\end{aligned}$$

We show that Assumption D4 implies that the second summand is $o_p\left(h^{\frac{P}{2}}\right)$ in Section D.3.6, Lemma D1 below, see also Horowitz (1992), Lemma 5. The first summand is $O_p\left(MSE\left(\hat{Y}\right)\right)$ because

$$\begin{aligned}
&\left[\int\int \left(K\left(\frac{\hat{Y}_t(u,v) - y_t}{h}\right) - K\left(\frac{Y_t(u,v) - y_t}{h}\right)\right)^2 dudv\right]^{1/2} = \\
&\left[\int\int \left(K'\left(\frac{Y_t(u,v) - y_t}{h}\right)\left[\frac{\hat{Y}_t(u,v) - Y_t(u,v)}{h}\right]\right)^2 dudv\right]^{1/2} + s_N
\end{aligned}$$

where s_N is asymptotically negligible since K is differentiable. The claim follows since P is chosen in Assumption D4 so that $h^{P/2}$ is $o_p\left(MSE\left(\hat{Y}\right)\right)$. $\square$

D.4.4 Inference on the bounds on the DPO and DTE

Proposition D2: Under Assumptions D1, D2, and D4

$$P\left(\left|\sum_{r\in\mathbb{N}}\hat{\lambda}_{rt}\hat{\lambda}_{rt'}-\sum_{r\in\mathbb{N}}\lambda_{rt}\lambda_{rt'}\right|>\epsilon\right)\leq P\left(\left(\xi_t'A_t\xi_t\right)^{1/2}F_{t'}(y_{t'})^{1/2}+\left(\xi_{t'}'A_{t'}\xi_{t'}\right)^{1/2}F_t(y_t)^{1/2}>\epsilon\right)+o_p(1)$$

where $A_t=V_t^{1/2}\Omega_tV_t^{1/2}$, $\Omega_t=\int\int\frac{1}{h^2}K'\left(\frac{Y_t(u,v)-y_t}{h}\right)^2X_t(u,v)'X_t(u,v)dudv$, and $(\xi_1,\xi_0)\sim\mathcal{N}\left(0,I_{dim(\beta_1)+dim(\beta_0)}\right)$.

Proof of Proposition D2: From the proof of Proposition D1 in Section D.3.3, Assumptions D1 and D3 imply that

$$\begin{aligned}\left|\sum_{r\in\mathbb{N}}\hat{\lambda}_{rt}\hat{\lambda}_{rt'}-\sum_{r\in\mathbb{N}}\lambda_{rt}\lambda_{rt'}\right|\leq&\left[\int\int\left(K'\left(\frac{Y_t(u,v)-y_t}{h}\right)\left[\frac{\hat{Y}_t(u,v)-Y_t(u,v)}{h}\right]\right)^2dudv\right]^{1/2}F_{t'}(y_{t'})^{1/2}\\&+\left[\int\int\left(K'\left(\frac{Y_{t'}(u,v)-y_{t'}}{h}\right)\left[\frac{\hat{Y}_{t'}(u,v)-Y_{t'}(u,v)}{h}\right]\right)^2dudv\right]^{1/2}F_t(y_t)^{1/2}+o_p(1).\end{aligned}$$

The claim then follows from Assumption D2, since

$$\begin{aligned}&\int\int\left(K'\left(\frac{Y_t(u,v)-y_t}{h}\right)\left[\frac{\hat{Y}_t(u,v)-Y_t(u,v)}{h}\right]\right)^2dudv\\=&\left(\hat{\beta}_t-\beta_t\right)'\left[\int\int\frac{1}{h^2}K'\left(\frac{Y_t(u,v)-y_t}{h}\right)^2X_t(u,v)'X_t(u,v)dudv\right]\left(\hat{\beta}_t-\beta_t\right)\to_d\xi_t'A_t\xi_t.\ \square\end{aligned}$$

Once can make inferences about $\sum_{r\in\mathbb{N}}\lambda_{rt}\lambda_{rt'}$ and the bounds on the DPO and DTE in practice by replacing A_t with the estimator $\hat{A}=\hat{V}_t^{1/2\prime}\hat{\Omega}_t\hat{V}_t^{1/2}$ where $\hat{V}_t$ is a consistent estimator of V_t, $\hat{\Omega}_t=\int\int\frac{1}{h^2}K'\left(\frac{\hat{Y}_t(u,v)-y_t}{h}\right)^2X_t(u,v)'X_t(u,v)dudv$, and $\hat{F}_t(y_t)=\int\int\mathbb{1}\{\hat{Y}_t(u,v)\leq y_t\}dudv$. One can use the right-hand side to construct critical values or confidence intervals in the usual way. Replacing $\frac{1}{h^2}K'\left(\frac{\hat{Y}_t(u,v)-y_t}{h}\right)^2$ with $\sup_x|\frac{1}{h^2}K'\left(\frac{x}{h}\right)^2|$ and $\hat{F}_{t'}(y_{t'})$ with 1 allows for inferences that are uniformly valid over $y_t,y_{t'}\in\mathbb{R}$.

D.4.5 Consistent estimation of the distribution of spectral treatment effects

Let $\{\sigma_{rt}\}_{r\in[R]}$ and $\{\hat{\sigma}_{rt}\}_{r\in[R]}$ denote the R largest eigenvalues of Y_t and $\hat{Y}_t$ in absolute value, ordered to be decreasing. We propose the estimator $S\hat{T}E(u,v;\phi) := \sum_{r\in[R]} (\hat{\sigma}_{r1} - \hat{\sigma}_{r0})\, \phi_r(u)\phi_r(v)$ for $STE(u,v;\phi) = \sum_{r\in[R]} (\sigma_{r1} - \sigma_{r0})\, \phi_r(u)\phi_r(v)$ and $\int\int K\left(\frac{S\hat{T}E(u,v;\phi)-y}{h}\right) dudv$ for its marginal $F_{STE}(y;\phi) := \int\int \mathbb{1}\{STE(u,v;\phi) \le y\}$.

Proposition D3: Under Assumptions D1 and D3

$$\sup_{y\in\mathbb{R}} \left| \int\int K\left(\frac{S\hat{T}E(u,v;\phi)-y}{h}\right) dudv - \int\int \mathbb{1}\{STE(u,v;\phi) \le y\} dudv \right| = o_p(1).$$

Proof of Proposition D3: Write

$$\begin{aligned}
&\left| \int\int K\left(\frac{S\hat{T}E(u,v;\phi)-y}{h}\right) dudv - \int\int \mathbb{1}\{STE(u,v;\phi) \le y\} dudv \right| \\
&\le \left| \int\int K\left(\frac{S\hat{T}E(u,v;\phi)-y}{h}\right) dudv - \int\int K\left(\frac{STE(u,v;\phi)-y}{h}\right) dudv \right| \\
&\qquad + \left| \int\int K\left(\frac{STE(u,v;\phi)-y}{h}\right) dudv - \int\int \mathbb{1}\{STE(u,v;\phi) \le y\} dudv \right|.
\end{aligned}$$

We show that Assumption D3 implies that the second summand is $o_p\left(h^P\right)$ in Section D.3.6 below, see also Horowitz (1992)'s Lemma 5. The first summand is $o_p(1)$ because

$$\begin{aligned}
&\left| \int\int K'\left(\frac{STE(u,v;\phi)-y}{h}\right) \left[\frac{S\hat{T}E(u,v;\phi) - STE(u,v;\phi)}{h}\right] dudv \right| \\
&= \left| \sum_{r\in[R]} \left((\hat{\sigma}_{r1} - \hat{\sigma}_{r0}) - (\sigma_{r1} - \sigma_{r0}) \right) W_r \right| \le \left[\sum_{s\in\{0,1\}} ||\hat{Y}_s - Y_s||_2 \right] \left(\sum_{r\in[R]} W_r^2 \right)^{1/2}
\end{aligned}$$

plus an asymptotically negligible term, where $W_r = \int\int \frac{1}{h} K'\left(\frac{STE(u,v;\phi)-y}{h}\right) \phi_r(u)\phi_r(v) dudv$ and $\sum_r W_r^2$ is finite because K' is square integrable by assumption. The claim follows since $h^P \to 0$ by Assumption D3. $\square$

D.4.6 Inference on the distribution of spectral treatment effects

Proposition D4: Under Assumptions D1-D3

$$P\left(\left|\int\int K\left(\frac{S\hat{T}E(u,v;\phi)-y}{h}\right)dudv-\int\int \mathbb{1}\{STE(u,v;\phi)\leq y\}dudv\right|>\epsilon\right)$$
$$\leq P\left(\left[(\xi_1' B_1\xi_1)^{1/2}+(\xi_0' B_0\xi_0)^{1/2}\right]\left(\sum_r W_r^2\right)^{1/2}>\epsilon\right)$$

where $W_r=\int\int \frac{1}{h}K'\left(\frac{STE(u,v;\phi)-y}{h}\right)\phi_r(u)\phi_r(v)dudv$,
$B_t=V_t^{1/2}\left[\int\int X_t(u,v)'X_t(u,v)dudv\right]V_t^{1/2}$, and $(\xi_1\ \xi_0)\to\mathcal{N}\left(0,I_{dim(\beta_1)+dim(\beta_0)}\right)$.

Proof of Proposition D4: From the proof of Proposition D3 in Section D.3.5, Assumptions D1 and D3 imply that

$$\left|\int\int K\left(\frac{S\hat{T}E(u,v;\phi)-y}{h}\right)dudv-\int\int \mathbb{1}\{STE(u,v;\phi)\leq y\}dudv\right|$$
$$\leq\left[\sum_{s\in\{0,1\}}||\hat{Y}_s-Y_s||_2\right]\left(\sum_{r\in[R]}W_r^2\right)^{1/2}+o_p\left(MSE\left(\hat{Y}\right)\right).$$

The claim then follows from Assumption D2, since

$$\int\int\left(\hat{Y}_t(u,v)-Y_t(u,v)\right)^2dudv=\left(\hat{\beta}_t-\beta_t\right)'\left[\int\int X_t(u,v)'X_t(u,v)\right]\left(\hat{\beta}_t-\beta_t\right)\to_d \xi_t' B_t\xi_t.\ \square$$

One can make inferences about $\int\int \mathbb{1}\{STE(u,v;\phi)\leq y\}dudv$ in practice by replacing W_r with the estimator $\hat{W}_r=\frac{1}{h}\int\int K'\left(\frac{S\hat{T}E(u,v;\phi)-y}{h}\right)\phi_r(u)\phi_r(v)dudv$ and B_t with $\hat{B}_t=\hat{V}_t^{1/2}\left[\int\int X_t(u,v)'X_t(u,v)dudv\right]\hat{V}_t^{1/2}$. One can use the right-hand side to construct critical values or confidence intervals in the usual way.

D.4.7 Additional lemmas

Lemma D1: Under Assumption D4,

$$\left[\int\int\left(K\left(\frac{Y_t(u,v)-y_t}{h}\right)-\mathbb{1}\{Y_t(u,v)\leq y_t\}\right)^2 dudv\right]=o_p\left(h^P\right).$$

Proof of Lemma D1: Write

$$\int\int\left(K\left(\frac{Y_t(u,v)-y_t}{h}\right)-\mathbb{1}\{Y_t(u,v)\leq y_t\}\right)^2 dudv$$
$$=\int\int K\left(\frac{Y_t(u,v)-y_t}{h}\right)^2 dudv+F_t(y_t)-2\int\int K\left(\frac{Y_t(u,v)-y_t}{h}\right)\mathbb{1}\{Y_t(u,v)\leq y_t\}dudv.$$

The first summand

$$\begin{aligned}\int\int K\left(\frac{Y_t(u,v)-y_t}{h}\right)^2 dudv &= -2\int K(\tau)K'(\tau)F_t(y_t+h\tau)d\tau\\ &= -2\int K(\tau)K'(\tau)\left[F_t(y_t)+f_t(y_t)h\tau+...+\frac{1}{P!}f_t^P(y_t)h^P\tau^P+o_p\left(h^P\right)\right]d\tau\\ &= F_t(y_t)+o_p\left(h^P\right)\end{aligned}$$

where the first equality is due to a change in variables and integration by parts, the second equality is Taylor's Theorem, and the last equality is by the choice of K in Assumption D4: $2\int K(\tau)K'(\tau)d\tau=-1$ and $\int K(\tau)K'(\tau)\tau^p d\tau=0$ for $p\in[P]$.

Similarly, the third summand

$$\begin{aligned}\int\int K\left(\frac{Y_t(u,v)-y_t}{h}\right)\mathbb{1}\{Y_t(u,v)\leq y_t\}dudv &= \int K'(\tau)\mathbb{1}\{\tau\leq 0\}F_t(y_t+h\tau)d\tau\\ &= \int K'(\tau)\mathbb{1}\{\tau\leq 0\}\left[F_t(y_t)+f_t(y_t)h\tau+...+\frac{1}{P!}f_t^P(y_t)h^P\tau^P+o_p\left(h^P\right)\right]d\tau\\ &= o_p\left(h^P\right)\end{aligned}$$

where the last equality is also by the choice of K in Assumption D4: $\int_{\tau\leq 0}K'(\tau)\tau^P d\tau=0$ for $p=0$ and $p\in[P]$. The claim follows. $\square$

Lemma D2: Under Assumption D3,

$$\left| \int\int K\left(\frac{STE(u,v;\phi)-y}{h}\right) dudv - \int\int \mathbb{1}\{STE(u,v;\phi)\leq y\} dudv \right| = o_p\left(h^P\right).$$

Proof of Lemma D2: Write

$$\begin{aligned}\int\int K\left(\frac{STE(u,v;\phi)-y}{h}\right) dudv &= \int K'(\tau)F_t(y_t+h\tau)d\tau \\ &= \int K'(\tau)\left[F_t(y_t)+f_t(y_t)h\tau+...+\frac{1}{P!}f_t^P(y_t)h^P\tau^P+o_p\left(h^P\right)\right]d\tau \\ &= F_t(y_t)+o_p\left(h^P\right)\end{aligned}$$

where the first equality is due to a change in variables and integration by parts, the second equality is Taylor's Theorem, and the last equality is by the choice of K in Assumption D3: $\int K'(\tau)d\tau = -1$ and $\int K'(\tau)\tau^p d\tau = 0$ for $p \in [P]$. □

References

Athey, Susan, Dean Eckles, and Guido W Imbens, "Exact p-values for network interference," *Journal of the American Statistical Association*, 2018, *113* (521), 230–240.

Auerbach, Eric, "Identification and estimation of a partially linear regression model using network data," *Econometrica*, 2022, *90* (1), 347–365.

_ , "Testing for Differences in Stochastic Network Structure," *Econometrica*, 2022, *90* (3), 1205–1223.

Badev, Anton, "Discrete Games in Endogenous Networks: Equilibria and Policy," *arXiv preprint arXiv:1705.03137*, 2017.

Bai, Jushan, Serena Ng et al., "Large dimensional factor analysis," *Foundations and Trends® in Econometrics*, 2008, *3* (2), 89–163.

Bajari, Patrick, Brian Burdick, Guido W Imbens, Lorenzo Masoero, James McQueen, Thomas Richardson, and Ido M Rosen, "Multiple Randomization Designs," *arXiv preprint arXiv:2112.13495*, 2021.

Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou, "Who's who in networks. wanted: the key player," *Econometrica*, 2006, *74* (5), 1403–1417.

Banerjee, Abhijit, Arun G Chandrasekhar, Esther Duflo, and Matthew O Jackson, "The diffusion of microfinance," *Science*, 2013, *341* (6144), 1236498.

Basse, Guillaume, Peng Ding, Avi Feller, and Panos Toulis, "Randomization tests for peer effects in group formation experiments," *arXiv preprint arXiv:1904.02308*, 2019.

Basse, Guillaume W, Avi Feller, and Panos Toulis, "Randomization tests of causal effects under interference," *Biometrika*, 2019, *106* (2), 487–494.

Birkhoff, Garrett, "Three observations on linear algebra," *Univ. Nac. Tacuman, Rev. Ser. A*, 1946, *5*, 147–151.

Birman, Michael Sh and Michael Z Solomjak, *Spectral theory of self-adjoint operators in Hilbert space*, Vol. 5, Springer Science & Business Media, 2012.

Bonhomme, Stéphane and Elena Manresa, "Grouped patterns of heterogeneity in panel data," *Econometrica*, 2015, *83* (3), 1147–1184.

Bramoullé, Yann and Garance Genicot, "Diffusion centrality: Foundations and extensions," 2018.

Calvó-Armengol, Antoni, Eleonora Patacchini, and Yves Zenou, "Peer effects and social networks in education," *The Review of Economic Studies*, 2009, *76* (4), 1239–1267.

Chatterjee, Sourav, "Matrix estimation by universal singular value thresholding," *The Annals of Statistics*, 2015, *43* (1), 177–214.

Comola, Margherita and Silvia Prina, "Treatment effect accounting for network changes," *Review of Economics and Statistics*, 2021, *103* (3), 597–604.

Cruz, Cesi, Julien Labonne, and Pablo Querubin, "Politician family networks and electoral outcomes: Evidence from the Philippines," *American Economic Review*, 2017, *107* (10), 3006–37.

Dudley, Richard M, *Real analysis and probability*, Vol. 74, Cambridge University Press, 2002.

Dzemski, Andreas, "An empirical model of dyadic link formation in a network with unobserved heterogeneity," *Review of Economics and Statistics*, 2019, *101* (5), 763–776.

Finke, Gerd, Rainer E Burkard, and Franz Rendl, "Quadratic assignment problems," in "North-Holland Mathematics Studies," Vol. 132, Elsevier, 1987, pp. 61–82.

Graham, Bryan S, "An econometric model of network formation with degree heterogeneity," *Econometrica*, 2017, *85* (4), 1033–1063.

_ , "Network data," in "Handbook of Econometrics," Vol. 7, Elsevier, 2020, pp. 111–218.

Hardy, Godfrey, John Littlewood, and George Pólya, *Inequalities*, Cambridge university press, 1952.

Hodges, Joseph L and Erich L Lehmann, "Estimates of location based on rank tests," in "Selected Works of EL Lehmann," Springer, 2012, pp. 287–300.

Hoffman, AJ and HW Wielandt, "The variation of the spectrum of a normal matrix," *Duke Math J*, 1953, *20*, 37–39.

Horowitz, Joel L, "A smoothed maximum score estimator for the binary response model," *Econometrica*, 1992, pp. 505–531.

Jackson, Matthew O and Asher Wolinsky, "A strategic model of social and economic networks," *Journal of economic theory*, 1996, *71* (1), 44–74.

Jochmans, Koen, "Two-way models for gravity," *Review of Economics and Statistics*, 2017, *99* (3), 478–485.

_ **and Martin Weidner**, "Fixed-effect regressions on network data," *Econometrica*, 2019, *87* (5), 1543–1560.

Lehmann, Erich L and Joseph P Romano, *Testing statistical hypotheses*, Springer Science & Business Media, 2006.

Leung, Michael P, "Two-step estimation of network-formation models with incomplete information," *Journal of Econometrics*, 2015, *188* (1), 182–195.

Mele, Angelo, "A structural model of Dense Network Formation," *Econometrica*, 2017, *85* (3), 825–850.

Menzel, Konrad, "Strategic network formation with many agents," 2015.

Ridder, Geert and Shuyang Sheng, "Estimation of Large Network Formation Games," 2015.

Rosenbaum, P.R., *Observational Studies*, Springer, 2002.

Stock, James H and Mark W Watson, "Dynamic factor models, factor-augmented vector autoregressions, and structural vector autoregressions in macroeconomics," in "Handbook of macroeconomics," Vol. 2, Elsevier, 2016, pp. 415–525.

Thirkettle, Matthew, "Identification and estimation of network statistics with missing link data," *Unpublished manuscript*, 2019.

Whitt, Ward, "Bivariate distributions with given marginals," *The Annals of statistics*, 1976, *4* (6), 1280–1289.